# ON THE INTERSECTION OF TWO CONICS HAVING A COMMON FOCUS.

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THE problem of constructing the common points or tangents of two conics, determined by a number of given points or tangents, is, in general, of the fourth degree, and consequently beyond the reach of the machinery allowed in elementary geometry. When two common points or tangents are given, the remaining common elements may be constructed by means of a ruler and completely given conic; but even here, the actual construction is often laborious. In the special cases indicated in the title of this paper, there happens to be a very simple construction by means of ruler and compass.

Suppose that $C_1$ be the common focus of two conics, and that we have besides at least three points or tangents of each given. Take any circle $c_1^2$ with its centre at $C_1$. The polar reciprocal of each of these conics with regard to $c_1^2$ is, of course, a circle. The pole or polar of any line or point with regard to $c_1^2$ may be constructed by means of ruler and compass, hence we may construct three points or tangents to each circle reciprocal to our given conics. The circles themselves may then be constructed and, by elementary geometry, their common tangents. The poles of these lines should then be found, and these will be the intersections required. It should be noticed that, if our conics are given by points, we shall need four points of each to make our construction unique, for we see in the reciprocal case, that four circles may be constructed tangent to three given lines.

An interesting application of the foregoing construction, appears in connection with the problem of Appolonius: *to construct a circle tangent to three given circles.* Let the circles be called $c_1^2$, $c_2^2$, $c_3^2$, their centres $C_1$, $C_2$, $C_3$ and radii $r_1$, $r_2$, $r_3$. For the sake of definiteness, we will suppose that each lies wholly without the others, while $r_1 > r_2$ and $r_1 > r_3$. If $C$ be the centre of a circle tangent externally to $c_1^2$ and $c_2^2$, or tangent to and surrounding both, we see that $C\,C_1 - C\,C_2 = r_1 - r_2$. The locus of such points is, then, a hyperbola with one focus at $C_1$ and the other at $C_2$. We may construct as many points $C$ as we please; for if $r$ be any convenient length we merely have to find the intersection of a circle with centre $C_1$ and radius $r_1 + r$, with one having centre $C_2$ and radius $r_2 + r$. This will be one of the required points $C$.

A like form of construction will give points on a hyperbola having an analogous relation to the circles $c_1^2, c_3^2$. The two hyperbolas will have the common focus $C_1$, hence their intersections may be found by the construction outlined above. There will be four of these points, but, as we shall presently see, two must be rejected; the other two will be centres of circles fulfilling the given condition.*

The problem of the number of solutions may be notably simplified by the consideration that there are just as many solutions in the general case, as in the special case where the centres of the given circles all lie on a straight line. For a circle with its centre at the point of concurrence of the radical axes, and radius equal to the tangent thence to any one of them, will cut all three orthogonally. If we take a centre of inversion on this new circle at a point without all three of the given circles, and invert the whole figure, we shall get three new circles cutting the same straight line orthogonally, that is, having collinear centres. Circles touching all three of the original circles, will invert into others touching all the new ones, and conversely. No one of the new circles will surround another, for, as we move along the line of centres, we shall not intersect one circle, while tracing a diameter of another. We may use the old notation without confusion, calling the new circle which lies between the other two $c_1^2$ and noticing that the former inequality existing between the radii may fall away. We may, as before, find two hyperbolas, with the common focus $C_1$; and we shall see that they are turned in opposite ways with regard to $C_1$. They will then meet in four real points, on the same branch of one curve, and in pairs on the two branches of the other. Two of these must be rejected as they will be points whence circles may be drawn tangent externally to one pair of circles and internally to the other. The other two will be points sought. There will be, besides these hyperbolas, two others. One will be the locus of centres of circles touching $c_1^2$ externally and $c_2^2$ internally, or *vice versa*; the other will be the analogous curve for $c_1^2\ c_3^2$. Each pair of hyperbolas will give two solutions, so that the total number, when the given circles lie exterior to one another, is eight.

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* The idea of solving the Appolonian problem by means of the intersection of hyperbolas is certainly not new. Adrianus Romanus tried to carry out such a solution. *Cf.* Cajori. *A History of Mathematics* p. 154, New York, 1884.